

## The Numerical Solution of a Volterra Integral Equation\*

GORDON KENT

*Physical Electronics Laboratory, Department of Electrical Engineering,  
Syracuse University, Syracuse, New York 13210*

AND

JOSEPH MAUTZ

*Department of Electrical Engineering, Syracuse University, Syracuse, New York 13210*

Received February 5, 1968

### ABSTRACT

A method is presented for the numerical solution of the Volterra integral equation which describes signal propagation on an electron beam or electrostatic oscillations of a temperate plasma. The method is based on a step function approximation of the velocity distribution function, an approximation suitable for direct inclusion of experimental data, and Laplace transform analysis is used to obtain a solution expressed as an almost periodic function. Results of computation are presented for an example typical of the electron beam problem. It is shown that for a specified maximum error the order of the approximation of the distribution function is a linearly increasing function of the maximum value of the independent variable.

### INTRODUCTION

The problem of signal propagation along a polychromatic electron beam [1] and the problem of electrostatic oscillations in a temperate plasma [2] are, in a large measure, formally identical [3]. The former is a boundary value problem and the latter is an initial value problem, but both can be stated in terms of the Volterra integral equation of the Faltung type,

$$y(x) = k(x) - i\lambda^2 \int_0^x k(x - \xi) y(\xi) d\xi \quad (1)$$

The formal solutions of these problems have been developed from the alternate techniques of Laplace transform [2] and Fourier analysis [4]. While the latter is mathematically more esoteric, requiring theorems from theories of generalized functions and singular integral equations, both methods have been shown to yield

---

\* Supported in part by National Science Foundation Grant G-21998.

equivalent results [5]. It is from these techniques, which have been extensively discussed in the literature, as well as from iterative methods [6] that one must choose a numerical procedure for the solution of Eq. (1).

The need for a numerical method stems from the interest in determining experimentally whether or not the physical assumptions from which (1) is derived are in fact valid for the electron beam problem [7, 8]. To settle the question, one must determine the kernel of (1) from experimental data, find a numerical solution, and test that solution against additional experimental data. Moreover, the method of solution should lend itself to the reverse procedure, i.e., determination of the solution from experimental data, solving the inverse equation to find the kernel, and testing this result against experimental data.

In this paper we describe a numerical procedure based on a simple approximation of the distribution function which appears in the kernel. The approximation is suitable for fitting experimental points, and the kernel realized is such that Laplace transform analysis is reduced to a matter of determining poles, zeros, and residues of a rational function. The solution obtained is in the form of a sum of a finite number of sinusoidal terms, the number depending on the degree of the approximation to the distribution function. It is, therefore, an almost periodic function, and it approximates the exact solution for a finite range of  $x$ . This range is roughly proportional to the number of terms in the sum. At least part of the exact solution is in the form of a Fourier integral; and if this part is to be approximated by an almost periodic function, the linear relationship between the range of the variable and the number of terms, for a given accuracy, is to be expected.

#### APPROXIMATION FORMULAS

For the electron beam the terms in (1) are defined as follows:

$f(\eta)$  = the longitudinal velocity distribution function, normalized so that

$$\int f(\eta) d\eta = 1, \quad (2)$$

$$\int \eta f(\eta) d\eta = 1, \quad (3)$$

$\eta$  = the longitudinal velocity, normalized with respect to the average unperturbed beam velocity;

$$k(x) = \int (df/d\eta) \exp(-ix/\eta) d\eta \quad (4)$$

= the ballistic current, normalized with respect to the product of the beam conductance and the modulation voltage [9];

- $y(x)$  = convection current, normalized as is  $k(x)$ ;  
 $x$  = distance, normalized with respect to  $(v/\omega)$ , where  $v$  is the average velocity and  $\omega$  the modulation frequency;  
 $\lambda = (\omega_p/\omega)$ , where  $\omega_p$  is the plasma frequency of the beam.

Equation (1) is based on the assumptions that (i) the modulation is a small perturbation of the dc beam velocities, (ii) that only the longitudinal coordinate is of consequence, and (iii) that only velocity modulation occurs.

The solution of (1) can be expected to be of the form

$$y(x) = w(x) \exp(-ix) \quad (5)$$

where  $w(x)$  is a relatively slowly varying function. Accordingly, it is convenient to transform (1) to an equation in  $w(x)$ , and one obtains by substitution of (5) into (1) the equation

$$w(x) = h(x) - i\lambda^2 \int_0^x h(x - \xi) w(\xi) d\xi, \quad (6)$$

in which the new kernel is given by

$$h(x) = \int (df/d\eta) \exp[-ix(1/\eta - 1)] d\eta. \quad (7)$$

Since the ultimate result desired is  $|y|^2$ , which, according to (5) is the same as  $|w|^2$ , we need be concerned only with the solution of (6).

With the aid of the convolution theorem, one can take the Laplace transform of (6) and solve for the transform of  $w(x)$  to obtain

$$W(p) = H(p)/[1 + i\lambda^2 H(p)] \quad (8)$$

where

$$\begin{aligned} W(p) &= \mathcal{L}[w(x)] \\ H(p) &= \mathcal{L}[h(x)]. \end{aligned}$$

The problem now is to find a representation of  $H(p)$  as a rational function so that the inversion of (8) is reduced in essence to the calculation of poles and corresponding residues.

The approximation of the distribution function and, hence, the kernel is more conveniently carried through after writing the kernel in a somewhat different form. The form is obtained by a partial integration with respect to  $\eta$  followed by the translation of the origin given by

$$\theta = (1/\eta - 1). \quad (9)$$

In this manner one gets

$$h(x) = -ix \int_{\phi_1}^{\phi_2} g(\theta) \exp(-i\theta x) d\theta, \quad (10)$$

where

$$g(\theta) = f[1/(1 + \theta)] \\ \phi_1 = (1/\eta_2 - 1); \quad \phi_2 = (1/\eta_1 - 1); \quad \eta_2 > \eta_1.$$

It is assumed here and throughout the following that  $f(\eta)$  is nonzero only when  $\eta_1 > \eta > \eta_2$ .

Let  $g_n(\theta)$  be the  $n$ 'th order approximation of  $g(\theta)$ , given by the sum of step functions

$$g_n(\theta) = \sum_{j=1}^{n+1} a_j S[\theta - (\phi_1 + (j-1)\Delta)] \quad (11)$$

where  $\Delta = (\phi_2 - \phi_1)/n$ . In order that the approximation vanish outside the end points, a necessary constraint on the coefficients is

$$\sum_{j=1}^{n+1} a_j = 0. \quad (12)$$

By substituting (11) into (10) and making use of (12), one obtains

$$h_n(x) = - \sum_{j=1}^{n+1} a_j \exp[-ix(\phi_1 + (j-1)\Delta)] \quad (13)$$

and the Laplace transform of (13) is

$$H_n(p) = - \sum_{j=1}^{n+1} \frac{a_j}{p + i[\phi_1 + (j-1)\Delta]}. \quad (14)$$

The choice of the  $(n+1)$   $a_j$ 's is a matter of curve fitting, and there are many criteria of goodness which might be adopted. Since the moments of the distribution function have some interpretation in terms of macroscopic, measurable physical quantities, one is tempted to employ a curve fitting procedure which makes the moments of the approximate function equal to the exact moments. It is apparent, however, that as  $n$  is increased, improving the approximation, the constraints introduced for this purpose become numerically inconsequential. This fact, in some cases at least, shows itself in the linear system of equations for the  $a_j$ 's; as  $n$  is increased, the matrix of the system becomes progressively more ill-conditioned. It is likely, therefore, that moment invariant curve fitting is computationally imprac-

tical. For the sake of completeness, however, two such curve fitting procedures are outlined in the appendix to this paper. Both serve as examples of the inherent difficulties described.

The simplest curve fitting procedure is to set

$$\sum_{j=1}^k a_j = b_k = g\left(\phi_1 + \left(k - \frac{1}{2}\right)\Delta\right) \quad k = 1, 2, \dots, n. \tag{15}$$

It has the advantage that when  $n$  is sufficiently large, the approximation oscillates about the actual function, and there is inconsequential violation by the approximation of the normalization conditions, (2) and (3). Expressions (15) and (12) constitute a system of  $(n + 1)$  linear equations for the  $a_j$ 's. Moreover, the solution can be obtained by inspection. It is

$$a_j = b_j - b_{j-1}, \quad j = 1, 2, \dots, (n + 1), \tag{16}$$

$$b_0 = b_{n+1} = 0.$$

While (11) is an approximation of the distribution function by step functions, we could have, alternatively but less palatably, approximated the derivative of the distribution function by a sum of delta functions. In the latter case, (16) would take the form

$$a_1 = g'(\phi_1)(\phi_2 - \phi_1)/2n,$$

$$a_j = g'(\phi_1 + (j - 1)\Delta)(\phi_2 - \phi_1)/n, \quad j = 2, 3, \dots, n, \tag{17}$$

$$a_{n+1} = g'(\phi_2)(\phi_2 - \phi_1)/2n.$$

Expression (17) shows the  $a_j$ 's decrease inversely as  $n$  increases.

The next step in the solution is the determination of the poles and corresponding residues of the transform  $W_n(p)$ . In view of (8) and (14) it is evident that  $W_n(p)$  is a rational function expressible as the ratio of two polynomials, and it has the following special properties:

- (i) The point at infinity is a double zero. This follows from the constraint (12).
- (ii) If all of the  $a_j$ 's are different from zero there are  $(n + 1)$  poles and  $(n - 1)$  zeros on the finite plane.
- (iii) If the sequence of  $a_j$ 's has only one sign change, all the finite zeros and poles lie on the imaginary axis.

In the following analysis it is assumed that all the  $a_j$ 's are nonvanishing and that the sequence has only one sign change. The first assumption is not particularly restrictive and can be removed at a later time. From the physics of the problem, there is little reason to expect that the latter assumption need be altered.

In view of the above properties,  $W_n(p)$  can be written in the form

$$W_n(p) = \sum_{k=1}^{n+1} \frac{A_k}{p + iq_k} \quad (18)$$

where  $(-iq_1, -iq_2, \dots, -iq_{n+1})$  are the poles of  $W_n(p)$  and  $(A_1, A_2, \dots, A_{n+1})$  are the corresponding residues. The approximate solution of (6) is the inverse transform of (18)

$$w_n(x) = \sum_{k=1}^{n+1} A_k \exp(-iq_k x) \quad (19)$$

and the form of the solution required is

$$\frac{1}{2} |w_n(x)|^2 = \frac{1}{2} \sum_{j,k}^{n+1} A_j A_k \exp i(q_k - q_j)x \quad (20)$$

since the  $A_j$ 's are all real. The solution may also be written

$$\frac{1}{2} |w_n(x)|^2 = \frac{1}{2} \sum_{j=1}^{n+1} A_j^2 + \sum_{j=2}^{n+1} A_j \sum_{k=1}^{j-1} A_k \cos(q_k - q_j)x. \quad (21)$$

Thus it is apparent that the next task is computation of the  $2(n+1)$  constants appearing in these results.

For the calculation of the poles and residues of  $W_n(p)$ , it is convenient to introduce the following notation:

$$p = -iq, \quad (22)$$

$$H_n(p) = iF_n(q). \quad (23)$$

Hence

$$F_n(q) = \sum_{j=1}^{n+1} \frac{a_j}{-q + (\phi_1 + (j-1)\mathcal{A})} \quad (24)$$

and

$$W_n(p) = iF_n(q)/(1 - \lambda^2 F_n(q)). \quad (25)$$

Evidently, the  $q_k$ 's appearing in expressions (18)–(21) are the  $(n+1)$  roots of the rational equation

$$F_n(q) = 1/\lambda^2. \quad (26)$$

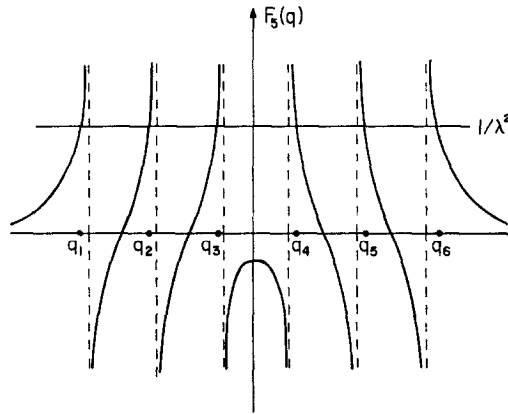


FIG. 1. Sketch of  $F_5(q)$ , showing alternate poles and zeros.

The understanding of the problem of solving (26) is facilitated by the example illustrated in Fig. 1. The sketch of  $F_5(q)$  shows six isolated poles and four zeros, which are also the zeros of  $W_5$ . A zero exists between adjacent poles except in the interval corresponding to the point where the sequence of  $a_j$ 's changes sign. The intersections of  $F_5(q)$  with the horizontal line  $(1/\lambda^2)$  are the six solutions of (26). For this example, it is evident that in general the roots of (26) satisfy the inequalities

$$\begin{aligned} q_1 < \phi_1 < q_2 < (\phi_1 + \Delta) < q_3 \cdots q_l < \phi_1 + (l - 1)\Delta, \\ q_{n+1} > \phi_2 > q_n > (\phi_2 - \Delta) \cdots q_{l+1} > \phi_1 + l\Delta, \end{aligned} \tag{27}$$

providing  $a_k > 0, k = 1, 2, \dots, l$ , and the remaining  $a_j$ 's are negative.

When  $\lambda^2$  is very small, the solutions of (26) occur near the poles of  $F_n(q)$ . Thus the first approximation of the  $k$ 'th solution is

$$q_k^{(0)} = \phi_1 + (k - 1)\Delta - a_k \lambda^2. \tag{28}$$

Improvement on this first estimate can be obtained by Newton's method. The procedure should converge rapidly because the derivative at this point is of the order of  $(1/\lambda^4)$ . If the superscript  $(m)$  indicates the order of the iteration, Newton's formula is

$$q_k^{(m)} = q_k^{(m-1)} + (1 - \lambda^2 F_n(q_k^{(m-1)})) / \lambda^2 F_n'(q_k^{(m-1)}). \tag{29}$$

The  $A_k$ 's are obtained from the formula

$$A_k = -1/\lambda^4 F_n'(q_k). \tag{30}$$

From this expression it is apparent that the  $A_k$ 's are essentially a byproduct of the computation of the  $q_k$ 's, when Newton's method is used. When (28) gives a reason-

able estimate of  $q_k$ , it follows from (30) that  $A_k$  is approximately proportional to  $a_k$ , and hence decreases linearly with  $n$ .

The total  $2(n + 1)$  constants appearing in (21) are not independent. There exist  $n$  relations between them so that the  $(n + 2)$  remaining constants are uniquely determined by the  $(n + 2)$  data points. Two of the  $n$  relations are simple enough to be useful as checks on the validity of the computation. These are:

$$\sum_{j=1}^{n+1} A_k = 0, \tag{31}$$

and

$$\sum_{k=1}^{n+1} q_k = (1/2)(n + 1)(\phi_1 + \phi_2). \tag{32}$$

EVALUATION OF THE METHOD

The proposed method was evaluated by comparing results for a particular example with results obtained by a presumably more exact technique.

The "exact" method depends on writing the transform of (7) in the form

$$H(-iq) = i \int_{-\phi}^{+\phi} \frac{g'(\theta)}{\theta - q} d\theta, \quad \text{Im } q > 0 \tag{33}$$

and choosing  $g(\theta)$  so that the indicated integration can be carried out explicitly. Expression (33) defines an analytic function on the complex  $q$  plane, cut on the straight line joining  $(-\phi, 0)$  to  $(\phi, 0)$ , and a corresponding branch cut is required to make  $W(p)$  single-valued. Thus the inversion integral for obtaining  $w(x)$  can be expressed as an integral around the branch cut and a summation over residues. Specifically, one gets

$$w(x) = - \int_{-\phi}^{+\phi} \frac{Q(q) \exp(-iqx)}{[1 - \lambda^2 F(q)]^2 + [\lambda^2 \pi Q(q)]^2} dq - \sum_{k=a,b} \frac{\exp(-iq_k x)}{\lambda^2 \frac{dF}{dq} \Big|_{q_k}}, \tag{34}$$

where

$$F(q) = \text{Principal Value} \int_{-\phi}^{+\phi} \frac{g'(\theta)}{\theta - q} d\theta,$$

$$Q(q) = \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0} \int_{-\phi}^{+\phi} g'(\theta) \left[ \frac{1}{\theta - (q + i\epsilon)} - \frac{1}{\theta - (q - i\epsilon)} \right] d\theta,$$

$q_a ; q_b$  are the zeros of  $1 - \lambda^2 F(q)$ .



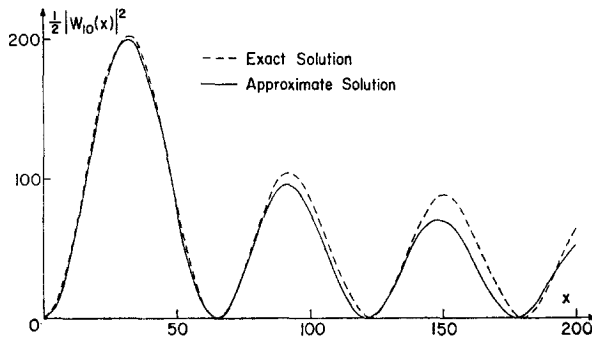


FIG. 2. Normalized power against normalized distance, as calculated for  $n = 10$  and by the standard method.

If  $g(\theta)$  is rational, the latter two defining integrals can be evaluated explicitly, and numerical integration of (34) yields the desired standard solution.

The example was specified by the choice

$$\begin{aligned}
 g(\theta) &= \frac{3}{4\phi} (1 + \theta)^3 (1 - \theta^2/\phi^2), & |\theta| \leq \phi \\
 &= 0, & |\theta| > \phi,
 \end{aligned}
 \tag{35}$$

where  $\phi$  was fixed to correspond to a velocity spread of 10%. The parameter  $\lambda^2$  was assigned the value  $10^{-3}$ . Both  $g(\theta)$  and the parameters are reasonably typical of the electron beam problem.

The numerical evaluation of the integral in (34) was accomplished by fitting a sum of step functions to the spectral density in the integrand followed by analytic

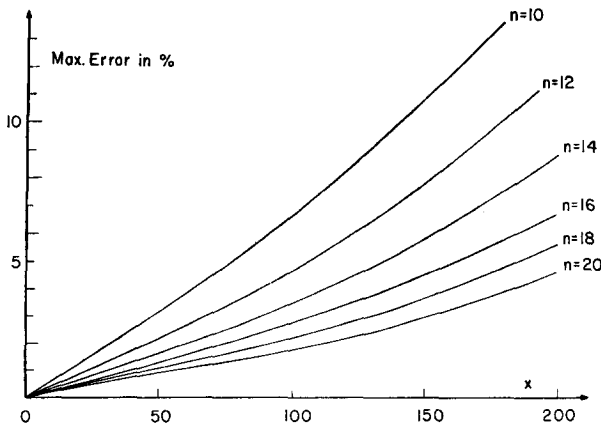


FIG. 3. Maximum error in the calculation against normalized distance, with  $n$  as a parameter.

integration. The fit was made at one hundred equally spaced points. The numbers  $q_a$  and  $q_b$  were obtained by Newton's method.

Figure 2 shows the results of the 10th-order approximation and the "exact" solution plotted against  $x$ . The two solutions agree rather well for  $x < 70$ , but a significant deviation in both amplitude and the location of zeros begins to develop for larger  $x$ .

If we define the error as the difference between these curves divided by the first peak value of the "exact" solution, the error is an oscillating function of  $x$ . Plots of the error against  $x$  were made, and envelopes of these oscillating curves were constructed graphically. These envelopes, shown in Fig. 3, represent the maximum possible error. It is clear from Fig. 2 that the maximum error occurs near but not necessarily at the value of  $x$  taken from Fig. 3. Figure 4 illustrates the nearly linear relationship between the order of approximation and the maximum value of  $x$  for a given maximum error.

For the example presented, the asymptotic solution consists of two space charge waves. The contribution of the integral in (34) is  $O(1/x)$  at least, according to the Riemann-Lebesgue lemma [10], and it goes to zero in the truly exact solution. Figures 5 and 6 show the calculations of the propagation constants and the amplitudes of these waves plotted against  $n$ . The isolated points are the estimates of these numbers obtained from the "exact" calculation. These two figures demonstrate the fact that separation of the solution into its transient and asymptotic parts results in large errors in each part, while the total solution is comparatively very good.

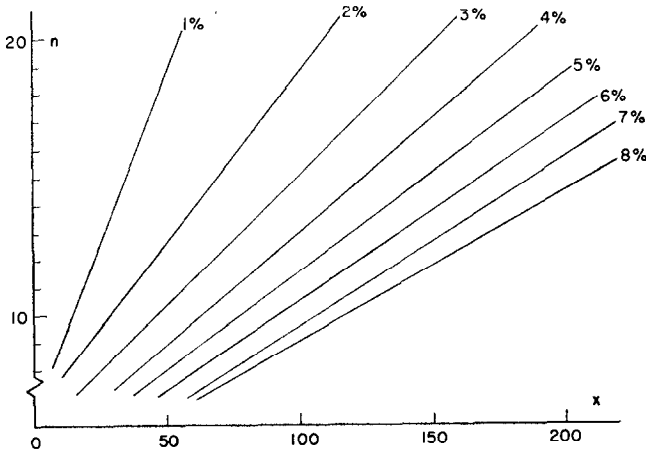


FIG. 4. Order of approximation against distance, with maximum error as a parameter.

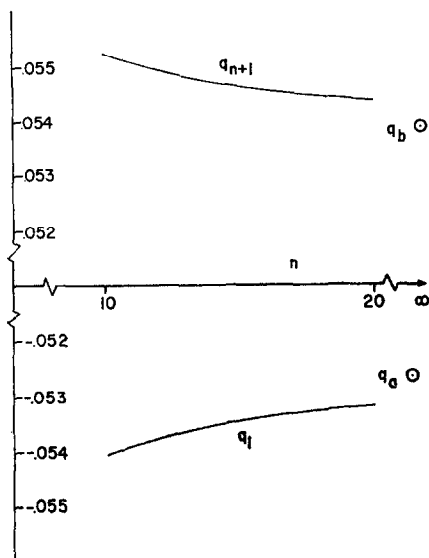


FIG. 5. Illustration of the phase error of the space charge waves as dependent on the order of approximation.

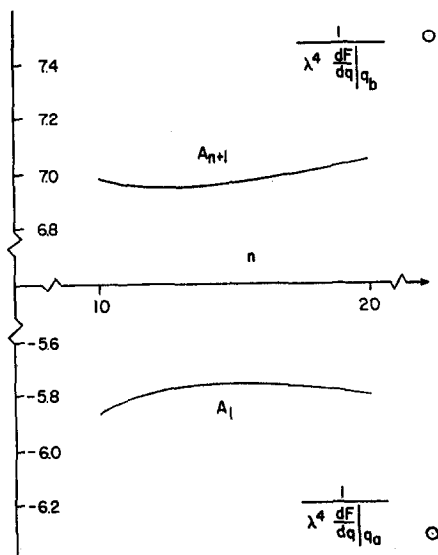


FIG. 6. Illustration of the amplitude error of the space charge waves as dependent on the order of approximation.

## CONCLUDING REMARKS

The point has been argued in some detail [3] that the experimental measurement of a distribution function with instruments limited by noise can produce only a finite, albeit optimum, number of data points. Such being the case, there is little justification for fitting the measured values of the distribution function in any way more elaborate than the step function approximation proposed here. The interpolation between data points will, of course, have a profound effect on the asymptotic solution, but this fact means simply that the asymptotic solution is no better than the sagacity of the guess at an interpolation rule. If one agrees that calculation of the asymptotic solution is not meaningful in view of experimental limitations, then the results of computation should be accepted only to values of  $x$  such that the error is insensitive to the method of interpolation.

In view of the significance of the analyticity of  $f'(\eta)$ , as discussed by Landau, Van Kampen, and many others, it is difficult to conceive of a worse interpolation rule than that used here. Nevertheless, the results compare favorably with the "exact" solution for limited values of  $x$ , and the legitimacy of a smoother interpolation is a moot question.

The "exact" calculation used as a standard of comparison suffers from the same limitations as the proposed method. Both solutions approximate a continuous spectrum as a discrete one, and both are expressed in terms of almost periodic functions. When the number of waves in the discrete spectrum is small, it is to be expected that errors in both the amplitude and the zero locations should appear at small values of  $x$ . Moreover, the part of the solution corresponding to the evanescent part of the truly exact solution will ultimately, at large  $x$ , begin to repeat itself, almost, and the error will approach 100%. This property of an almost periodic function [11] can be used to support, with more rigor than given here, the contention that there is a linear relationship between the order of approximation and the maximum value of  $x$  for a specified maximum error.

While this method makes it a simple matter to incorporate measurements of the distribution function into the calculations, it is not of much help in the inverse problem. A desirable solution to both problems together would consist of a method for approximating a continuous spectrum with a discrete one in such a way as to optimize the curve fitting to two sets of experimental data.

## ACKNOWLEDGMENTS

The authors are indebted to the Syracuse University Computing Center for computations and to Dr. Long-Fei Chang for advice and discussions.

APPENDIX. MOMENT-INVARIANT CURVE-FITTING

One moment-invariant curve-fitting procedure is the choice of the  $(n + 1) a_j$ 's to make a least-squares fit to the data, subject to the constraints imposed by (12) and the two normalization conditions, (2) and (3). The latter are expressed as

$$\int_{\phi_1}^{\phi_2} \frac{g_n(\theta)}{(1 + \theta)^2} d\theta = \sum_{j=1}^{n+1} \frac{a_j}{1 + \phi_1 + (j - 1)\Delta} = 1 \tag{A1}$$

corresponding to (2), and

$$\int_{\phi_1}^{\phi_2} \frac{g_n(\theta)}{(1 + \theta)^3} d\theta = \frac{1}{2} \sum_{j=1}^{n+1} \frac{a_j}{(1 + \phi_1 + (j - 1)\Delta)^2} = 1 \tag{A2}$$

corresponding to (3). If the data points at which the fit is to be made are the  $n$  constants

$$b_j = g(\phi_1 + (j - \frac{1}{2})\Delta), \quad j = 1, 2, \dots, n$$

then  $(b_j - \sum_{k=1}^j a_k)$  is the difference between the data and the approximation at the points  $\theta = \phi_1 + (j - \frac{1}{2})\Delta$ . The  $a_k$ 's are to be chosen so that the square of the difference summed over all  $j$ 's is a minimum, subject to the constraints (12), (A1), and (A2). The constraints require the introduction of three Lagrange multipliers  $(\lambda_1, \lambda_2, \lambda_3)$ , and the function to be extremized is

$$F(a_1, a_2, a_3, \dots, a_{n+1}) = \sum_{j=1}^{n+1} \left( b_j - \sum_{k=1}^j a_k \right)^2 - \lambda_1 \sum_{j=1}^{n+1} \frac{a_j}{1 + \phi_1 + (j - 1)\Delta} - \lambda_2 \frac{1}{2} \sum_{j=1}^{n+1} \frac{a_j}{(1 + \phi_1 + (j - 1)\Delta)^2} - \lambda_3 \sum_{j=1}^{n+1} a_j. \tag{A3}$$

It is to be understood in (A3) that  $b_{n+1}$  is zero.

Differentiation of (A3) with respect to  $a_l$  ( $l = 1, 2, \dots, n + 1$ ) and equating the derivatives to zero yields the following set of  $(n + 1)$  linear equations

$$\sum_{j=l}^{n+1} \sum_{k=1}^j a_k = \sum_{k=l}^{n+1} b_k + \frac{1}{2} \lambda_1 \frac{1}{1 + \phi_1 + (l - 1)\Delta} + \frac{1}{4} \lambda_2 \frac{1}{(1 + \phi_1 + (l - 1)\Delta)^2} + \frac{1}{2} \lambda_3, \quad l = 1, 2, \dots, n + 1. \tag{A4}$$

This result as it stands is not in convenient form for obtaining the  $a_i$ 's, but an

elementary transformation of this system gives the desired form. The left member of (A4) is the column vector  $A = (a_1, a_2, \dots, a_{n+1})$ , premultiplied by the matrix  $B$ ,

$$B = \{B_{jk}\} = \begin{bmatrix} n+1 & n & n-1 & \dots & 2 & 1 \\ n & n & n-1 & \dots & 2 & 1 \\ n-1 & n-1 & n-1 & \dots & 2 & 1 \\ \vdots & & & & & \\ 2 & 2 & 2 & \dots & 2 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix}.$$

This matrix is diagonalized by the elementary transformation

$$\delta_{jk} = \begin{cases} 2B_{jk} - (B_{j-1,k} + B_{j+1,k}) & j = 2, 3, \dots, n+1, \\ B_{1k} - B_{2k}, & j = 1, \end{cases}$$

where

$$\begin{aligned} \delta_{jk} &= 1, & j &= k \\ &= 0, & j &\neq k, \\ B_{n+2,k} &= 0. \end{aligned}$$

This transformation is equivalent to multiplying the  $l$ th equation of (A4) by 2 and subtracting the sum of the  $(l-1)$ st and  $(l+1)$ st equations for all  $l$  greater than one. For  $l=1$ , it is merely necessary to subtract the second equation from the first. The result is the following set of  $n$  equations.

$$\begin{aligned} a_1 &= b_1 + \frac{\lambda_1}{2} \left\{ \frac{1}{1 + \phi_1} - \frac{1}{1 + \phi_1 + \Delta} \right\} + \frac{\lambda_2}{4} \left\{ \frac{1}{(1 + \phi_1)^2} - \frac{1}{(1 + \phi_1 + \Delta)^2} \right\}, \\ a_l &= (b_l - b_{l-1}) + \frac{\lambda_1}{2} \left\{ \frac{2}{1 + \phi_1 + (l-1)\Delta} - \frac{1}{1 + \phi_1 + l\Delta} - \frac{1}{1 + \phi_1 + (l-2)\Delta} \right\} \\ &\quad + \frac{\lambda_2}{4} \left\{ \frac{2}{(1 + \phi_1 + (l-1)\Delta)^2} - \frac{1}{(1 + \phi_1 + l\Delta)^2} - \frac{1}{(1 + \phi_1 + (l-2)\Delta)^2} \right\}, \\ &\qquad\qquad\qquad l = 2, 3, \dots, n. \end{aligned} \tag{A5}$$

The  $(n+1)$ st equation defines  $\lambda_3$ , and the equation is not required except for this purpose.

The Lagrange multipliers can now be obtained by first using (12) to eliminate  $a_{n+1}$  from (A1) and (A2) and secondly using (A5) to eliminate the remaining  $a_j$ 's. The result of these substitutions is a set of two linear equations for the multipliers of the form

$$\begin{aligned} A_{11}\lambda_1 + A_{12}\lambda_2 &= C_1, \\ A_{21}\lambda_1 + A_{22}\lambda_2 &= C_2. \end{aligned} \tag{A6}$$

While this set of equations has a unique solution, both  $\lambda_1$  and  $\lambda_2$  increase in proportion to  $n$  when  $n$  is large. At the same time, however, the coefficients of the  $\lambda$ 's in (A5) decrease in proportion to  $(1/n^2)$ . Thus the contribution to the  $a_j$ 's of the constraints decreases as  $(1/n)$ , and the moment invariant feature of the curve fitting becomes insignificant. Nevertheless, this fact may not appear in the machine computation because of the loss of significant figures in solving (A6) and computing (A5).

A second curve fitting procedure gives an approximation with the first  $n$  moments of the approximate function equal to those of the exact function. This procedure yields, together with (12), the following system of  $(n + 1)$  equations:

$$\sum_{j=1}^{n+1} a_j = 0; \tag{A7}$$

$$\frac{1}{k + 1} \sum_{j=1}^{n+1} \frac{a_j}{[1 + \phi_1 + (j - 1)\Delta]^{k+1}} = P_k, \quad k = 0, 1, \dots, (n - 1),$$

where the  $P_k$ 's are the moments,

$$P_k = \int_{\eta_1}^{\eta_2} f(\eta) \eta^k d\eta. \tag{A8}$$

Expressions (A1) and (A2) are evidently included in (A7).

The practicality of this procedure depends on the sensitivity of the solution of (A7) to computational errors in the coefficients. The sensitivity increases as the magnitude of the determinant of the normalized coefficient matrix decreases relative to unity. This determinant can be written

$$\det(C_n) = \frac{1}{(\prod_{j=1}^{n+1} S_j)^{1/2}} \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^n \\ 1 & x_3 & x_3^2 & \dots & x_3^n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_{n+1} & x_{n+1}^2 & \dots & x_{n+1}^n \end{vmatrix}, \tag{A9}$$

where

$$x_j = 1/[1 + \phi_1 + (j - 1)\Delta],$$

$$S_j = 1 + x_j^2 + x_j^4 + \dots + x_j^{2n} = (x_j^{2n+2} - 1)/(x_j^2 - 1).$$

It is shown in various texts on numerical analysis (See, for example, Richard W. Hamming, "Numerical Methods for Scientists and Engineers," McGraw Hill,

New York, 1962) that the determinant in (A9) is the product of  $n(n+1)/2$  factors of the form  $(x_j - x_k)$ ,  $j \neq k$ , and hence (A9) can be written in the equivalent form

$$\det(C_n) = \frac{\prod_{j>k=1}^n (x_j - x_k)}{(\prod_{j=1}^{n+1} S_j)^{1/2}}. \quad (\text{A10})$$

This expression is convenient for obtaining an estimate, or at least a bounding relation, for the least upper bound of  $\det(C_n)$ . Since

$$1/(1 + \phi_1) = x_1 > x_2 > \cdots > x_{n+1} = 1/(1 + \phi_2), \quad (\text{A11})$$

it follows that

$$S_1 > S_2 > \cdots > S_{n+1}. \quad (\text{A12})$$

Thus

$$\det(C_n) < \frac{(x_1 - x_{n+1})^{n(n+1)/2}}{S_{n+1}^{(n+1)/2}}. \quad (\text{A13})$$

The term

$$1/S_{n+1} = 1 - \left[ \frac{(1 + \phi_2)^{2n} - 1}{(1 + \phi_2)^{2n+2} - 1} \right] < 1. \quad (\text{A14})$$

Assuming that  $(\phi_1 + \phi_2)$  is very small and that both  $\phi_1$  and  $\phi_2$  are less than one in magnitude.

$$(x_1 - x_{n+1}) = \frac{(\phi_2 - \phi_1)}{(1 + \phi_1)(1 + \phi_2)} \cong (\phi_2 - \phi_1). \quad (\text{A15})$$

Expressions (A13) and (A14) set upper bounds well above their minimum values, and one expects from their combination with (A15) the strong inequality

$$\det(C_n) \ll (\phi_2 - \phi_1)^{n(n+1)/2}. \quad (\text{A16})$$

Typical numbers for this particular problem are  $\phi_2 - \phi_1 = 0.2$  and  $n = 10$ . These give

$$\det(C_n) \ll (0.2)^{55} \cong 3 \times 10^{-39}. \quad (\text{A17})$$

One concludes from such numbers that the coefficient matrix is so extremely ill-conditioned that this curve fitting procedure is quite unworkable.



## REFERENCES AND FOOTNOTES

1. See, for example, J. BERGHAMMER, *J. Appl. Phys.* **3**, 4 (1962).
2. L. LANDAU, *J. Phys. U.S.S.R.* **10**, 25 (1946).
3. G. KENT, *Int. J. Electronics* **22**, 397-411 (1967).
4. N. G. VAN KAMPEN, *Physica* **21**, 949 (1955).
5. K. M. CASE, *Ann. Phys. (N.Y.)* **7**, 349 (1959).
6. See, for example, F. G. TRICOMI, "Integral Equations," Chap. 1. Interscience, New York (1963).
7. R. L. GUNSHOR, *J. Appl. Phys.* **37**, 1904 (1967).
8. M. CAULTON, *J. Appl. Phys.* **38**, 1839 (1967).
9. When the distribution function is discontinuous,  $df/d\eta$  is undefined, strictly speaking, but perturbation of the discontinuity is correctly included in the analysis if  $df/d\eta$  is interpreted as a function with the property that

$$\int (df/d\eta) \exp(-ix/\eta) d\eta = [f(\eta_0 + 0) - f(\eta_0 - 0)] \exp(-ix/\eta_0)$$

when integrated across the discontinuity at  $\eta_0$ .

10. E. T. WHITTAKER and G. N. WATSON, "Modern Analysis," (1948 ed. only). MacMillan, New York (1948).
11. A. S. BESICOVITCH, "Almost Periodic Functions." Dover, New York (1954).